Successive approximation method for solving (1+1)-dimensional dispersive long wave equations

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A B S T R A C T

In this paper, we study the (1+1)-dimensional dispersive long wave equations which describe the evolution of horizontal velocity component $u(x,t)$ of water waves of height $v(x,t)$, and solved it numerically by successive approximation method (SAM) to compare with Adomian's decomposition method (ADM), we found that SAM is suitable for this kind of problems also its effective and more accure than ADM. Mathematica has been used for computations.

1. Introduction

The celebrated (1+1)-dimensional dispersive long wave equations (Ablowitz and Clarkson, 1991; Broer, 1975)

$$u_t + v_x + \frac{1}{2}(u^2)_x = 0,$$
$$v_t + (uv + u + u_{xx})_x = 0. \tag{1}$$

Play important roles in nonlinear physics, which describe the evolution of horizontal velocity component $u(x,t)$ of water waves of height $v(x,t)$ propagating in both directions in an infinite narrow channel of finite constant depth. Many properties of (1) have been reported (Broer, 1975; Kaup, 1975; Kupershmidt, 1985).

It is interesting to study the extensions of (1) in higher-dimensional spaces. To date, there exist two prototypical extensions of (1) to cover the situation of wide channel or open seas. Boiti et al. (1987) presented the following (2+1)-dimensional extension related to (1)

$$u_{yt} + v_{xx} + \frac{1}{2}(u^2)_{xy} = 0,$$
$$v_t + (uv + u + u_{xy})_x = 0, \tag{2}$$

in the one-dimensional reduction $u = u(x + y, t)$, $v = v(x + y, t)$, system (2) reduces to system (1) that describes the travel of the shallow water wave.

Eckhaus (1985) presented another different two-dimensional extension of (1)

$$u_t + v_x + \frac{1}{2}(u^2)_x = 0,$$
$$v_{tx} + (uv + u + u_{xx})_{xx} + u_{yy} = 0, \tag{3}$$

which was obtained in the appropriate approximation from the basic equations of hydrodynamics. It is easy to see that if one makes the transformation either $u = u(x + y, t)$ and $v = v(x + y, t) – 1$ or $u = u(x + y, t)$ and $v = v(x + y, t)$, then (3) can also reduce to (1). Therefore, it follows that these two systems (2) and (3) can both reduce to the same system (1) under the proper transformations. But as Boiti et al. (1987) pointed out, system (2) is different from system (3).

2. Solitary wave solution

The exact solitary wave solution of the (2 + 1)-dimensional dispersive long wave equations (2) are:

$$u(x, y, t) = \alpha + \alpha \tanh\left(\frac{\alpha x + \beta y - \alpha^2 t + \delta}{2}\right), \tag{4}$$

$$v(x, y, t) = -1 + \frac{\alpha\beta}{2}\operatorname{sech}^2\left(\frac{\alpha x + \beta y - \alpha^2 t + \delta}{2}\right) \tag{5}$$

where α, β and δ are arbitrary constants.

In particular, taking $\beta = \alpha$ and setting $x + y = z$ in system (1), we obtain a special solitary wave solution of (1 + 1)-dimensional dispersive long wave equations (1),

$$u(z, t) = \alpha + \alpha \tanh\left(\frac{\alpha z - \alpha^2 t + \delta}{2}\right), \tag{6}$$

$$v(z, t) = -1 + \frac{\alpha^2}{2}\operatorname{sech}^2\left(\frac{\alpha z - \alpha^2 t + \delta}{2}\right), \tag{7}$$

which was obtained by Bai et al. (2006) and Wang et al. (1996).

3. Basic idea of successive approximation method (SAM)

The method of SAM ([Jerri, 1999; Saeed, 2006](#)), provides a method that can, in principle, be used to solve any initial value problem

$$\frac{\partial u}{\partial t} = f(u,t), \quad u(z,t_0) = u_0. \tag{8}$$

It starts by observing that any solution to (8) must also be a solution to

$$u(z,t) = u_0 + \int_{t_0}^{t} f(u(z,s),s)\, ds, \tag{9}$$

and then iteratively constructs a sequence of solutions that can get closer and closer to the actual (exact) solutions of (9). The SAM is based on the integral equation (9), as follows:

$$u_0(z,t) = u_0,$$
$$u_1(z,t) = u_0 + \int_{t_0}^{t} f(u_0(z,s),s)\, ds,$$
$$u_2(z,t) = u_0 + \int_{t_0}^{t} f(u_1(z,s),s)\, ds,$$
$$u_3(z,t) = u_0 + \int_{t_0}^{t} f(u_2(z,s),s)\, ds.$$

This process can be continued to obtain the nth approximation,

$$u_n(z,t) = u_0 + \int_{t_0}^{t} f(u_{n-1}(z,s),s)\, ds, \quad n = 1,2,\dots .$$

Then determine whether $u_n(z,t)$ approaches the solution $u(z,t)$ as n increases. This is done by proving the following:
The sequence $\{u_n(z,t)\}$ converges to a limit $u(z,t)$, that is:

$$\lim_{n\to\infty} u_n(z,t) = u(z,t), \quad t_0 \le z \le t .$$

The limiting function $u(z,t)$ is a solution of (9) on the interval $t_0 \le z \le t$. The solution $u(z,t)$ of (9) is unique. A proof of these results can be constructed along the lines of the corresponding proof for ordinary differential equations ([Coddington, 1995](#)).

4. SAM applied to (1+1)-dimensional dispersive long wave equations

In this section, we solve the (1+1)-dimensional dispersive long wave equations:

$$\left.\begin{array}{l} \frac{\partial u(z,s)}{\partial s} = -\frac{\partial v(z,s)}{\partial z} - \frac{1}{2}\frac{\partial u^2(z,t)}{\partial z}, \\ \frac{\partial v(z,s)}{\partial s} = -\frac{\partial}{\partial z}\big(u(z,s) + u(z,s)\,v(z,s)\big) - \frac{\partial^3 u(z,s)}{\partial z^3}, \end{array}\right\} \tag{10}$$

with initial solutions

$$u(z,0) = u_0 \text{ and } v(z,0) = v_0,$$

by using SAM as follows:

Integrating both sides of equation (10) with respect to s, from 0 to t, we get

$$u(z,t) = u(z,0) - \int_0^t \left(\frac{\partial v(z,s)}{\partial z} + \frac{1}{2}\frac{\partial u^2(z,s)}{\partial z}\right) ds,$$
$$\left.v(z,t) = v(z,0) - \int_0^t \left(\begin{array}{c}\frac{\partial}{\partial z}\big(u(z,s) + u(z,s)\,v(z,s)\big) \\ + \frac{\partial^3 u(z,s)}{\partial z^3}\end{array}\right) ds.\right\} \tag{11}$$

Using the initial conditions in (11) we get:

$$u(z,t) = u_0 - \int_0^t \left(\frac{\partial v(z,s)}{\partial z} + \frac{1}{2}\frac{\partial u^2(z,s)}{\partial z}\right) ds,$$
$$\left.v(z,t) = v_0 - \int_0^t \left(\begin{array}{c}\frac{\partial}{\partial z}\big(u(z,s) + u(z,s)\,v(z,s)\big) \\ + \frac{\partial^3 u(z,s)}{\partial z^3}\end{array}\right) ds.\right\} \tag{12}$$

Start with substituting initial approximation $u_0(z,t)$ and $v_0(z,t)$ in the integral equation (12) to obtain a first approximation $u_1(z,t)$ and $v_1(z,t)$

$$u_1(z,t) = u_0 - \int_0^t \left(\frac{\partial v_0(z,s)}{\partial z} + \frac{1}{2}\frac{\partial u_0^2(z,s)}{\partial z}\right) ds,$$
$$\left.v_1(z,t) = v_0 - \int_0^t \left(\begin{array}{c}\frac{\partial}{\partial z}\big(u_0(z,s) + u_0(z,s)\,v_0(z,s)\big) \\ + \frac{\partial^3 u_0(z,s)}{\partial z^3}\end{array}\right) ds.\right\} \tag{13}$$

Then this $u_1(z,t)$ and $v_1(z,t)$ is substituted again in the integral of (12) after replacing t by s to obtain a second approximation $u_2(z,t)$ and $v_2(z,t)$,

$$u_2(z,t) = u_0 - \int_0^t \left(\frac{\partial v_1(z,s)}{\partial z} + \frac{1}{2}\frac{\partial u_1^2(z,s)}{\partial z}\right) ds,$$
$$v_2(z,t) = v_0 - \int_0^t \left(\begin{array}{c}\frac{\partial}{\partial z}\big(u_1(z,s) + u_1(z,s)\,v_1(z,s)\big) \\ + \frac{\partial^3 u_1(z,s)}{\partial z^3}\end{array}\right) ds.$$

This process can be continued to obtain the nth approximation

$$u_n(z,t) = u_0 - \int_0^t \left(\frac{\partial v_{n-1}(z,s)}{\partial z} + \frac{1}{2}\frac{\partial u_{n-1}^2(z,s)}{\partial z}\right) ds,$$
$$\left.v_n(z,t) = v_0 - \int_0^t \left(\begin{array}{c}\frac{\partial}{\partial z}\left(\begin{array}{c}u_{n-1}(z,s) \\ +u_{n-1}(z,s)\,v_{n-1}(z,s)\end{array}\right) \\ + \frac{\partial^3 u_{n-1}(z,s)}{\partial z^3}\end{array}\right) ds,\right\} \tag{14}$$

For $n =1,2,\dots.$ To solve the equation (14), we use the initial approximation $u_0(z,t)$ and $v_0(z,t)$ which are given in equation (6) and (7), respectively. Using the iteration formula (14), we can obtain $u_1(z,t), u_2(z,t), u_3(z,t),\ v_1(z,t), v_2(z,t)$ and $v_3(z,t)$ as follows:

$$u_1(z,t) = \alpha + \alpha\tanh\left(\frac{\alpha z+\delta}{2}\right) - \frac{\alpha^3}{2}\operatorname{sech}^2\left(\frac{\alpha z+\delta}{2}\right)t,$$
$$u_2(z,t) = \alpha + \alpha\tanh\left(\frac{\alpha z+\delta}{2}\right) - \frac{\alpha^3}{2}\operatorname{sech}^2\left(\frac{\alpha z+\delta}{2}\right)t -$$
$$\frac{\alpha^5}{4}\operatorname{sech}^2\left(\frac{\alpha z+\delta}{2}\right)\tanh\left(\frac{\alpha z+\delta}{2}\right)t^2 +$$
$$\frac{\alpha^7}{12}\operatorname{sech}^4\left(\frac{\alpha z+\delta}{2}\right)\tanh\left(\frac{\alpha z+\delta}{2}\right)t^3,$$
$$u_3(z,t) = \alpha + \alpha\tanh\left(\frac{\alpha z+\delta}{2}\right) - \frac{\alpha^3}{2}\operatorname{sech}^2\left(\frac{\alpha z+\delta}{2}\right)t -$$
$$\frac{\alpha^5}{4}\operatorname{sech}^2\left(\frac{\alpha z+\delta}{2}\right)\tanh\left(\frac{\alpha z+\delta}{2}\right)t^2 + \frac{\alpha^7}{24}\operatorname{sech}^2\left(\frac{\alpha z+\delta}{2}\right)\left(1 - \right.$$
$$\left.3\tanh^2\left(\frac{\alpha z+\delta}{2}\right)\right)t^3 - \frac{5\alpha^9}{192}\operatorname{sech}^4\left(\frac{\alpha z+\delta}{2}\right)\left(1 - \right.$$
$$\left.2\tanh\left(\frac{\alpha z+\delta}{2}\right) - 5\tanh^2\left(\frac{\alpha z+\delta}{2}\right) + \frac{5}{18}\tanh^3\left(\frac{\alpha z+\delta}{2}\right)\right)t^4 +$$

$\frac{\alpha^{11}}{240}\operatorname{sech}^4\left(\frac{\alpha z+\delta}{2}\right)\left(1-\frac{3}{2}\tanh\left(\frac{\alpha z+\delta}{2}\right)-8\tanh^2\left(\frac{\alpha z+\delta}{2}\right)+\frac{9}{2}\tanh^3\left(\frac{\alpha z+\delta}{2}\right)+7\tanh^4\left(\frac{\alpha z+\delta}{2}\right)\right)t^5 +$

$\frac{\alpha^{13}}{288}\operatorname{sech}^6\left(\frac{\alpha z+\delta}{2}\right)\tanh\left(\frac{\alpha z+\delta}{2}\right)\left(1-4\tanh^2\left(\frac{\alpha z+\delta}{2}\right)\right)t^6 -$
$\frac{\alpha^{15}}{2016}\operatorname{sech}^8\left(\frac{\alpha z+\delta}{2}\right)\tanh\left(\frac{\alpha z+\delta}{2}\right)\left(1-5\tanh^2\left(\frac{\alpha z+\delta}{2}\right)\right)t^7 ,$

$v_1(z,t)=-1+\frac{\alpha^2}{2}\operatorname{sech}^2\left(\frac{\alpha z+\delta}{2}\right)+$
$\frac{\alpha^4}{2}\operatorname{sech}^2\left(\frac{\alpha z+\delta}{2}\right)\tanh\left(\frac{\alpha z+\delta}{2}\right)t$

$v_2(z,t)=-1+\frac{\alpha^2}{2}\operatorname{sech}^2\left(\frac{\alpha z+\delta}{2}\right)+$
$\frac{\alpha^4}{2}\operatorname{sech}^2\left(\frac{\alpha z+\delta}{2}\right)\tanh\left(\frac{\alpha z+\delta}{2}\right)t-\frac{\alpha^6}{8}\operatorname{sech}^2\left(\frac{\alpha z+\delta}{2}\right)\left(1-3\tanh^2\left(\frac{\alpha z+\delta}{2}\right)\right)t^2+\frac{\alpha^8}{24}\operatorname{sech}^4\left(\frac{\alpha z+\delta}{2}\right)\left(1-5\tanh^2\left(\frac{\alpha z+\delta}{2}\right)\right)t^3 ,$

$v_3(z,t)=-1+\frac{\alpha^2}{2}\operatorname{sech}^2\left(\frac{\alpha z+\delta}{2}\right)+$
$\frac{\alpha^4}{2}\operatorname{sech}^2\left(\frac{\alpha z+\delta}{2}\right)\tanh\left(\frac{\alpha z+\delta}{2}\right)t-\frac{\alpha^6}{8}\operatorname{sech}^2\left(\frac{\alpha z+\delta}{2}\right)\left(1-3\tanh^2\left(\frac{\alpha z+\delta}{2}\right)\right)t^2-\frac{\alpha^8}{12}\operatorname{sech}^2\left(\frac{\alpha z+\delta}{2}\right)\tanh\left(\frac{\alpha z+\delta}{2}\right)\left(2-3\tanh^2\left(\frac{\alpha z+\delta}{2}\right)\right)t^3+\frac{\alpha^{10}}{192}\operatorname{sech}^4\left(\frac{\alpha z+\delta}{2}\right)\left(5+35\tanh\left(\frac{\alpha z+\delta}{2}\right)-52\tanh^2\left(\frac{\alpha z+\delta}{2}\right)-75\tanh^3\left(\frac{\alpha z+\delta}{2}\right)+63\tanh^4\left(\frac{\alpha z+\delta}{2}\right)\right)t^4-\frac{3\alpha^{12}}{320}\operatorname{sech}^4\left(\frac{\alpha z+\delta}{2}\right)\left(3+\right.$

$40\tanh\left(\frac{\alpha z+\delta}{2}\right)-42\tanh^2\left(\frac{\alpha z+\delta}{2}\right)-152\tanh^3\left(\frac{\alpha z+\delta}{2}\right)+63\tanh^4\left(\frac{\alpha z+\delta}{2}\right)+112\tanh^5\left(\frac{\alpha z+\delta}{2}\right)\right)t^5 +$
$\frac{\alpha^{14}}{576}\operatorname{sech}^6\left(\frac{\alpha z+\delta}{2}\right)\left(1-19\tanh^2\left(\frac{\alpha z+\delta}{2}\right)+36\tanh^4\left(\frac{\alpha z+\delta}{2}\right)\right)t^6-\frac{\alpha^{16}}{4032}\operatorname{sech}^8\left(\frac{\alpha z+\delta}{2}\right)\left(1-24\tanh^2\left(\frac{\alpha z+\delta}{2}\right)+55\tanh^4\left(\frac{\alpha z+\delta}{2}\right)\right)t^7 .$

5. Results and discussion

We solved the (1+1)-dimensional dispersive long wave equations by SAM. Compared between 3rd-order approximate solutions of SAM and ADM (Mohamed, 2010) with exact solitary solutions. The results it shown by data and surfaces see in Tables 1a, 1b, 2a, and 2b and Figs. 1, 2, and 3. And we found the effects for α and δ on the solution of (1+1)-dimensional dispersive long wave equations as shown in Figs. 4a, 4b, 5a, and 5b. It shown that α has effects on long and short in wave and δ has effects on move wave only.

Table 1a: Exact solution, 3rd-order approximation solutions ADM (Mohamed, 2010) and SAM of $u(z,t)$ when $\alpha = 2$, $\delta = 0$ and $z = -0.01$

t	Exact	ADM	SAM
0	1.98000066664	1.98000066664	1.98000066664
0.01	1.9400179935224	1.9400179965071	1.9400178614665
0.02	1.9000832500842	1.9000833114428	1.9000811646242
0.03	1.8602282193671	1.860228598649	1.8602178003544
0.04	1.8204844305057	1.8204858453275	1.8204519384
0.05	1.7808830595711	1.7808870386801	1.7808047973543
0.06	1.7414548327879	1.7414641659086	1.7412947478293
0.07	1.7022299327534	1.7022492142149	1.7019374154519
0.08	1.6632379082584	1.6632741708006	1.662745783697
0.09	1.6245075882634	1.6245710228678	1.6237302965644
0.1	1.5860670005411	1.5861717576181	1.584898961109

Table 1b: Exact solution, 3rd-order approximation solutions ADM (Mohamed, 2010) and SAM of $v(z,t)$ when $=2$, $\delta = 0$ and $z = -0.01$

t	Exact	ADM	SAM
0	0.99980001333258	0.99980001333258	0.99980001333258
0.01	0.99820107944946	0.99820086649064	0.99820111308927
0.02	0.99500832154314	0.99500491883283	0.99500884803683
0.03	0.99023192466936	0.9902147296339	0.99023454277574
0.04	0.98388708015456	0.98383285816862	0.98389524331417
0.05	0.97599388320855	0.97586186371178	0.97601363254549
0.06	0.96657719825563	0.96630430553815	0.96661796371
0.07	0.95566649352597	0.95516274292249	0.95574201102401
0.08	0.9432956467829	0.94243973513958	0.94342503665934
0.09	0.92950272436413	0.9281378414642	0.92971177325638
0.1	0.91432973597948	0.9122596211711	0.91465242115369

Table 2a: Absolute error between exact solution and 3rd-order approximation solutions ADM (Mohamed, 2010) and SAM of $u(z,t)$ when $\alpha = 2$, $\delta = 0$ and $z = -0.01$

t	| Exact - ADM |	| Exact - SAM |
0	0	0
0.01	2.9847E-09	1.32056E-07
0.02	6.13586E-08	2.08546E-06
0.03	3.79282E-07	1.0419E-05
0.04	1.41482E-06	3.24921E-05
0.05	3.97911E-06	7.82622E-05
0.06	9.33312E-06	0.000160085
0.07	1.92815E-05	0.000292517
0.08	3.62625E-05	0.000492125
0.09	6.34346E-05	0.000777292
0.1	0.000104757	0.001168039
Total Error	0.000238906	0.003013449

Table 2b: Absolute error between exact solution and 3rd-order approximation solutions ADM (Mohamed, 2010) and SAM of $v(z,t)$ when $\alpha = 2$, $\delta = 0$ and $z = -0.01$

t	| Exact - ADM |	| Exact - SAM |
0	0	0
0.01	2.12959E-07	3.36398E-08
0.02	3.40271E-06	5.26494E-07
0.03	1.7195E-05	2.61811E-06
0.04	5.4222E-05	8.16316E-06
0.05	0.000132019	1.97493E-05
0.06	0.000272893	4.07655E-05
0.07	0.000503751	7.55175E-05
0.08	0.000855912	0.00012939
0.09	0.001364883	0.000209049
0.1	0.002070115	0.000322685
Total Error	0.005274605	0.000808498

Exact u

Exact v

(a) **(b)**
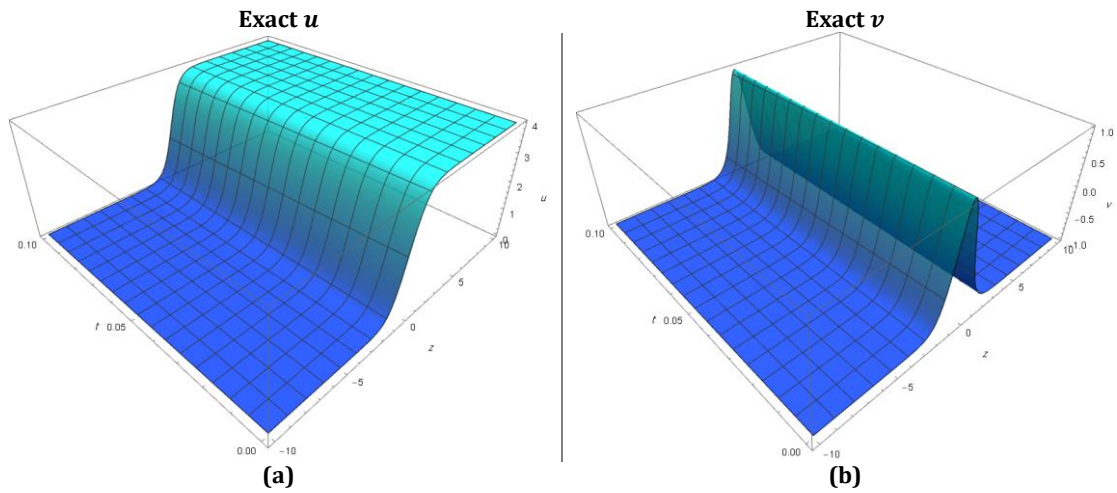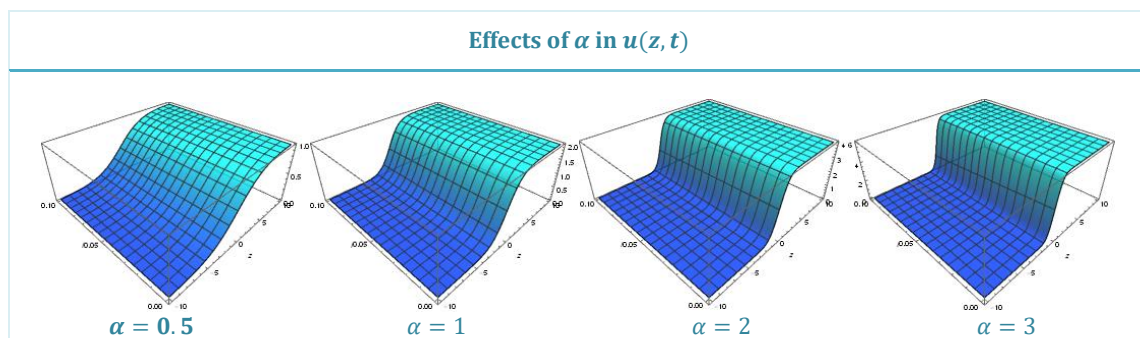
Fig. 1: The surfaces of exact solutions for $u(z,t)$ and $v(z,t)$, when $\alpha = 2$, $\delta = 0$ and $z \in [-10,10]$ at $0 \leq t \leq 0.1$

ADM u

ADM v

(a) **(b)**
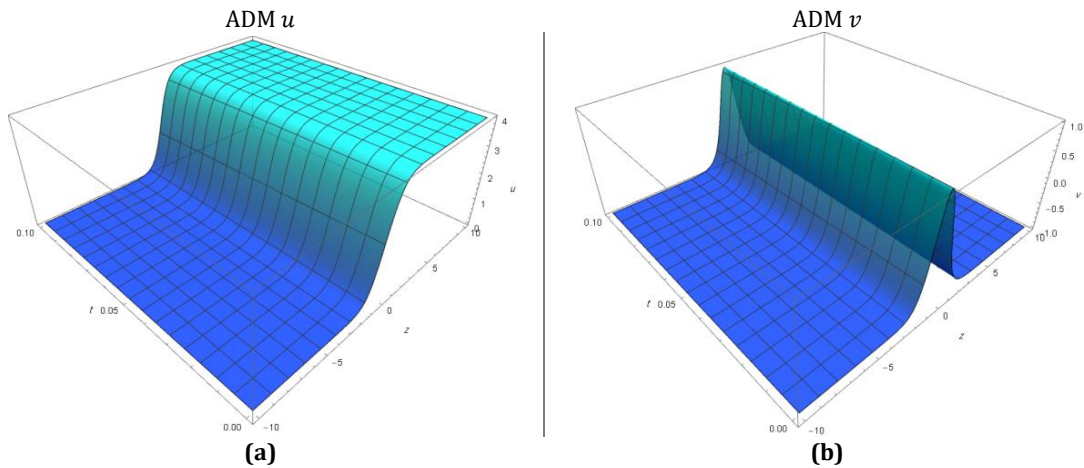
Fig. 2: The surfaces of ADM solutions for $u(z,t)$ and $v(z,t)$, when $\alpha = 2$, $\delta = 0$ and $z \in [-10,10]$ at $0 \leq t \leq 0.1$

SAM u

SAM v

(a) **(b)**
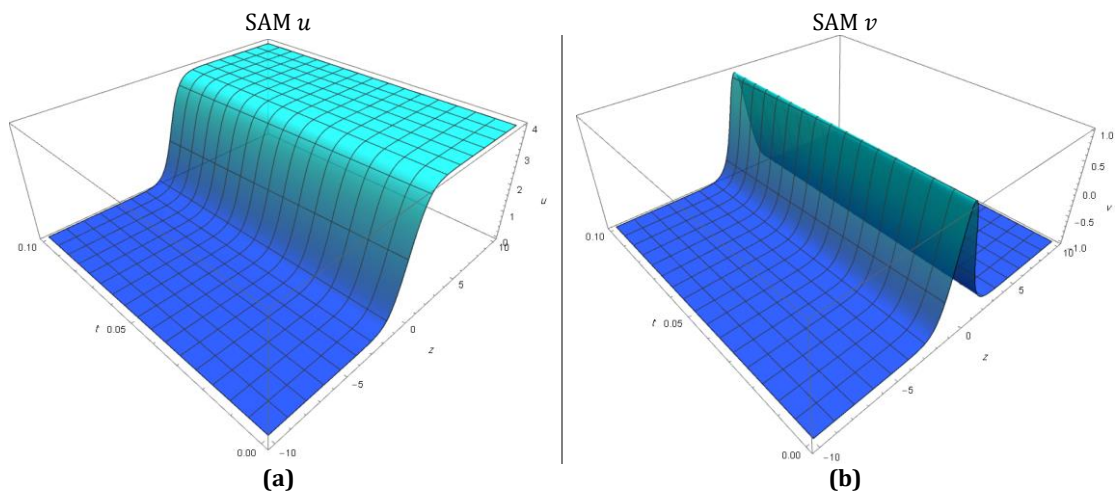
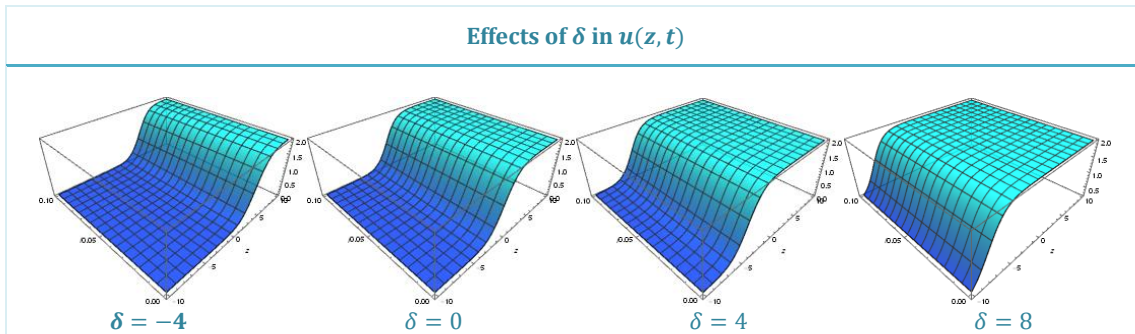**Fig. 3:** The surfaces of SAM solutions for $u(z,t)$ and $v(z,t)$, when $\alpha = 2$, $\delta = 0$ and $z \in [-10,10]$ at $0 \leq t \leq 0.1$

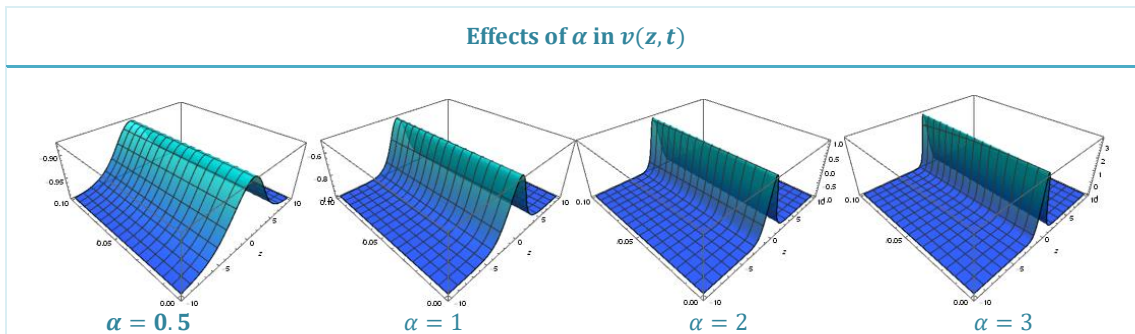**Effects of α in $u(z,t)$**

$\alpha = 0.5$ $\alpha = 1$ $\alpha = 2$ $\alpha = 3$

Fig. 4a: The surfaces of exact solution of $u(z,t)$, when $\alpha = 0.5, 1, 2, 3$ at $\delta = 0$, $z \in [-10,10]$, $0 \leq t \leq 0.1$

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Effects of δ in $u(z,t)$

$\delta = -4$ $\delta = 0$ $\delta = 4$ $\delta = 8$

Fig. 4b: The surfaces of exact solution of $u(z,t)$, when $\delta = -4, 0, 4, 8$ at $\alpha = 1, z \in [-10,10], 0 \leq t \leq 0.1$

Effects of α in $v(z,t)$

$\alpha = 0.5$ $\alpha = 1$ $\alpha = 2$ $\alpha = 3$

Fig. 5a: The surfaces of exact solution of $v(z,t)$, when $\alpha = 0.5, 1, 2, 3$ at $\delta = 0, z \in [-10,10], 0 \leq t \leq 0.1$

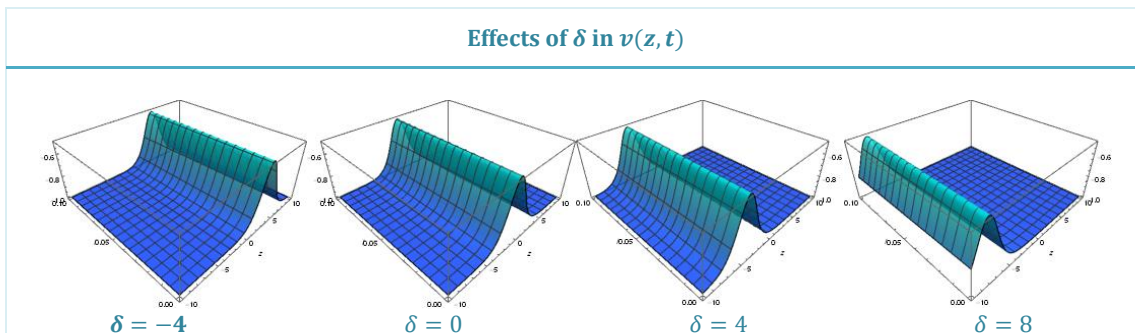**Effects of δ in $v(z,t)$**

$\delta = -4$ $\delta = 0$ $\delta = 4$ $\delta = 8$

Fig. 5b: The surfaces of exact solution of $v(z,t)$, when $\delta = -4, 0, 4, 8$ at $\alpha = 1, z \in [-10,10], 0 \leq t \leq 0.1$

6. Conclusion

In this paper, the SAM successfully applied to obtain approximate solutions of the (1+1)-dimensional dispersive long wave. We showed that successive approximation method easier, faster and more accurate than Adomain's decomposition method as shown especially in Tables 1b and 2b and Figs. 2b and 3b. Finally, it has become clear that α and δ have effects on solutions.

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List of symbols

SAM	Successive Approximate Method
ADM	Adomain Decomposition Method
Greek symbols	
α, β and δ	Constants

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